Lecture 32

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1 Powers of diagonalizable matrices

In this section we will give 2 algorithms of computing the m-th power of a matrix.

1.1 Method 1

First method is based on diagonalization. Suppose A is a given matrix, and we want to find its m-th power, i.e. we want to get a formula for A^m. We will suppose that the matrix A is diagonalizable. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A, and $e_1, e_2, \dots, e_n$ be its linearly independent eigenvectors. Then we know, that there exists a matrix C, whose columns are eigenvectors, and a diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

such that

$$D = C^{-1}AC, \quad \text{or} \quad A = CDC^{-1}.$$

Now we can see that

$$\begin{aligned} A^m &= (CDC^{-1})^m \\ &= \underbrace{(CDC^{-1})(CDC^{-1})\cdots(CDC^{-1})}_{m} \\ &= CD(C^{-1}C)D(C^{-1}\dots C)DC^{-1} \\ &= CDID\dots IDC^{-1} \\ &= CD^mC^{-1}. \end{aligned}$$

But

$$D^m = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}^m = \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n^m \end{pmatrix}$$

So,

$$A^m = C \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n^m \end{pmatrix} C^{-1}.$$

Example 1.1. *Let's find the formula for* $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^m$. *The characteristic polynomial is*

$$p_A(\lambda) = \lambda^2 - 3\lambda + 2.$$

So, the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$. *Let's compute eigenvectors.*

$\boldsymbol{\lambda_1 = 1}$. *After subtraction* $\lambda_1 = 1$ *from the diagonal, we have* $\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$, *so the eigenvector is* $(1, 1)$.

$\boldsymbol{\lambda_2 = 2}$. *After subtraction* $\lambda_2 = 2$ *from the diagonal, we have* $\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}$, *so the eigenvector is* $(2, 1)$.

Thus,

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Now we get

$$\begin{aligned} A^m &= CD^mC^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^m \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^m \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2^{m+1} \\ 1 & 2^m \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 + 2^{m+1} & 2 - 2^{m+1} \\ -1 + 2^m & 2 - 2^m \end{pmatrix}. \end{aligned}$$

1.2 Method 2

Let A be an $n \times n$-matrix. Suppose it has n different eigenvalues. Then the algorithm goes as following. Let's write the **approximation equation**:

$$a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \dots + a_1 = t^m$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be different eigenvalues of A. Now let's substitute λ_i's into the approximation equation instead of t. We will have n different equations with n variables. So, we can find $a_{n-1}, a_{n-2}, \ldots, a_0$. Now, we will have that

$$A^m = a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 I.$$

Let's note that this formula involves $n-1$ different powers of a matrix.

Example 1.2. *Let again* $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$. *The approximation equation is*

$$at + b = t^m.$$

For A we already computed eigenvalues: $\lambda_1 = 1, \lambda_2 = 2$. Substituting them, we have:

$$\begin{cases} a + b = 1 \\ 2a + b = 2^m \end{cases}$$

Multiplying the first equation by 2 and subtracting from the second one, we get $-b = 2^m - 2$, so $b = 2 - 2^m$, and $a = 1 - b = 2^m - 1$. So,

$$A^m = (2^m - 1)\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} + (2 - 2^m)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{m+1} - 1 & 2 - 2^{m+1} \\ 2^m - 1 & 2 - 2^m \end{pmatrix}$$

So, this method, of course, gave the same result as the first one.

2 Powers of nondiagonalizable matrices

We'll consider 2×2-matrices.

In the last chapter we considered a method of computing the power of the diagonalizable matrix. Now we will generalize method 2 from the previous lecture to nondiagonalizable matrices. We used an approximation equation,

$$at + b = t^m$$

and substituted different eigenvalues to it to determine the coefficients. If we have just one eigenvalue, we will take the derivative of the approximation equation:

$$a = mt^{m-1},$$

and substitute eigenvalue to it. We'll demonstrate this approach in the following example.

Example 2.1. *Let* $A = \begin{pmatrix} 2 & -4 \\ 1 & 6 \end{pmatrix}$. *Its eigenvalue is equal to 4. The approximation equation is*

$$at + b = t^m,$$

and substituting $t = 4$*, we get*

$$4a + b = 4^m.$$

Now, taking the derivative of this equation we have

$$a = mt^{m-1},$$

and substituting $t = 4$*, we get*

$$a = m4^{m-1}.$$

So, $a = m4^{m-1}$*, and* $b = 4^m - m4^m$*. So, the formula for the* $m-$*-th power of* A *is*

$$\begin{aligned} A^m &= (m4^{m-1})A + (4^m - m4^m)I \\ &= \begin{pmatrix} 2m4^{m-1} + 4^m - m4^m & -m4^m \\ m4^{m-1} & 6m4^{m-1} + 4^m - m4^m \end{pmatrix} \end{aligned}$$

3 Square roots of diagonalizable matrices

In the previous chapters we saw how to compute m-th power of a diagonalizable matrix using eigenvectors and eigenvalues. Now we will consider a problem of finding a square root of a matrix. Suppose the matrix A is diagonalizable, i.e. it has n linearly independent eigenvectors $e_1, e_2, \dots, e_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Now if C is a matrix, where e_i's are written as columns, and D is a diagonal matrix with λ_i's over diagonal, then

$$A = CDC^{-1}.$$

Let all λ_i's be nonnegative numbers. Now let's consider a matrix $\sqrt{D}$ which has either positive or negative square roots of λ_i's on diagonal:

$$\sqrt{D} = \begin{pmatrix} \pm\sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \pm\sqrt{\lambda_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \pm\sqrt{\lambda_n} \end{pmatrix}$$

We can easily check that for such defined $\sqrt{D}$, we have $(\sqrt{D})^2 = D$. So we see that a square root of a matrix A can be obtained from D and C by the following formula:

$$C\sqrt{D}C^{-1}.$$

Let's denote that there are more than one square root of a matrix — and all of them can be obtained by choosing different signs before $\sqrt{\lambda_i}$'s in D^{-1}.

Example 3.1. *Let's compute a square root of* $A = \begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix}$*. The characteristic polynomial is* $p_A(\lambda) = \lambda^2 - 13\lambda + 36$*, so eigenvalues are* $\lambda_1 = 4, \lambda_2 = 9$*. Eigenvector, corresponding to* $\lambda_1 = 4$

is determined from equation $3x_1 + 2x_2 = 0$*, so it can be* $(2, -3)$*. Eigenvector, corresponding to* $\lambda_2 = 9$ *is determined from equation* $-2x_1 + 2x_2 = 0$*, so it can be* $(1, 1)$*. So,*

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}, \quad C^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}.$$

Now,

$$\sqrt{D} = \begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 3 \end{pmatrix}.$$

So, for positive signs in D *we have:*

$$\begin{aligned} \sqrt{A} &= \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 4 & 3 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 13 & 2 \\ 3 & 12 \end{pmatrix}. \end{aligned}$$

In the same way we can get other square roots (changing signs in D*).*

If A has negative eigenvalues, then this algorithm is not applicable, but in this case there are no square roots of A.